

Violation of Bell inequalities in larger Hilbert spaces: robustness and challenges

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We explore the challenges posed by the violation of Bell-like inequalities by d -dimensional systems exposed to imperfect state-preparation and measurement settings. We address, in particular, the limit of high-dimensional systems, naturally arising when exploring the quantum-to-classical transition. We show that, although suitable Bell inequalities can be violated, in principle, for any dimension of given subsystems, it is in practice increasingly challenging to detect such violations, even if the system is prepared in a maximally entangled state. We characterize the effects of random perturbations on the state or on the measurement settings, also quantifying the efforts needed to certify the possible violations in case of complete ignorance on the system state at hand.

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I. INTRODUCTION

Year 2014 has marked the 50th anniversary of John S. Bell's paper [1] on the Einstein-Podolski-Rosen paradox [2], a landmark achievement that has shaped the research in modern quantum mechanics. Since then, Bell inequalities, which have proven the incompatibility of quantum mechanics with any local realistic theory, have been established as a viable tool for deepening our understanding of the nature of reality and, from a more pragmatic perspective, identify unambiguously correlations of a non-classical nature [3].

Non-locality tests for discrete-variable systems have been formulated, over the years, based on both dichotomic and multichotomic local measurement settings [4], showing that bipartite d -level entangled states are in conflict with local realistic assumptions more strongly than entangled qubits. Inequalities for bipartite d -dimensional systems have been put forward [5], while various formulation for multipartite $d = 2$ systems have been proposed [6], and extended to generic arbitrary dimensional systems [7]. Continuous-variable Bell-like inequalities have been derived for bipartite states using the phase-space formalism [8], and extended to multipartite scenarios recently [9].

The investigation on non-locality of discrete large- d systems is particularly relevant in light of the goal of inferring non-classical effects in increasingly large systems. The possibility to unambiguously signal non-classical correlations, strong enough to violate a Bell-like inequality, would provide undoubtable evidence of the relevance of quantum theory in the large-scale domain, and bring new perspectives to the investigation of the quantum-to-classical transition [10].

First steps along similar lines of investigation have been made by addressing the class of states dubbed as micro-macro ones [11], which have then been extended to macro-macro scenarios [12].

When tackling the phenomenology of non-classical correlations in large-scale systems, the detrimental effects of improper measurements should be considered on the same footing as non-ideal state preparation. It has been shown that coarse-grained measurements do indeed affect the chances of falsifying local realistic models, even when dealing with otherwise ideal state resources [13]. On the other hand, a typical assessment of the robustness of Bell-like tests is based on the amount of white noise that a given resource state is able to tolerate before losing any discrepancy with local hidden variable (LHV) models [5]. This approach was the basis for the conclusion drawn by Kaszilowski *et al.* on bipartite d -level systems [4]. However, while such analyses provide useful information on the effects that imperfections and environmental mechanisms have on the revelation of non-classicality, a systematic study of the scaling behavior of non-idealities against the *size* of a given system has only been partially addressed [14–16]. Yet, this is key information for the quest of observing non-classicality at the large-scale limit. For instance, as we address in this paper, Bell inequalities formulated for large- d systems require a large number of measurement settings: their inadequate arrangement could have rather significant implications for the successful falsification of LHV models even when states endowed with a large degree of quantum correlations are considered. In this sense, our study goes well beyond currently available literature [4].

Our study here addresses precisely this important issue, which we tackle in a careful quantitative manner. By addressing experimentally relevant imperfections in the Bell test of d -dimensional systems, we point out, through a careful statistical analysis, the existence of thresholds in the amount of

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randomness that a given resource state (or, equivalently, measurement setting) can tolerate. As we find out, such threshold, which scales unfavorably with the dimensions of the local systems being used, is rather severe and poses significant questions on the suitability of Bell-like tests in systems verging to the mesoscopic or macroscopic scale, when evaluated against the actual value taken by d . In some conditions, even a maximally entangled two-qutrit state could fail to violate a Bell inequality built from slightly improper measurement settings.

The remainder of this paper is organized as follows. In Sec. II we introduce the inequalities addressed in this work, setting the notation that will be used throughout our analysis. Sec. III is devoted to the analysis of the robustness of such Bell-like tests for systems of increasing dimensionality and exposed to random perturbations of the otherwise ideal measurement settings required by each of the inequalities that we study. In Sec. IV we extend our study to the case of random initial states, showing the inherent difficulties related to running Bell tests on such resources, and discussing the effectiveness of more sophisticated statistical sampling techniques for ascertaining non-locality in randomly picked states of d -dimensional systems. Finally, in Sec. V we draw our conclusions.

II. GENERALIZED BELL INEQUALITIES

In this Section we review the formulation of Bell inequalities for d -dimensional systems that are used hereafter. The original formulation of the Bell inequalities relates the joint probabilities for local measurements with two possible outcomes ($d = 2$), typically the polarization of photons or a spin-1/2 system (dimension d and the spin quantum number l obey the relation $d = 2l + 1$). However, it is possible to extend (in a non unique way) such formulation to systems with higher local dimension [5]: We consider a coupled bipartite system and two parties (Alice and Bob) each of them having access to a single partition, of dimension d , of the system (either subsystem \mathcal{A} or \mathcal{B}) and able to measure two independent observables, A_1, A_2 and B_1, B_2 , respectively. The operators A s and B s are assumed to have the same spectrum, $A_i|A_i^\ell\rangle = \ell|A_i^\ell\rangle$ and $B_j|B_j^\ell\rangle = \ell|B_j^\ell\rangle$, with $\ell = 0, \dots, d-1$. Clearly, for d -dimensional subsystems the measurement can have d different outcomes. The generalized Bell inequalities are obtained summing the probabilities that Alice and Bob obtain equal (or correlated) outcomes of their measurements. In particular, the first Bell expression we will consider reads as

$$I \equiv P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1), \quad (1)$$

where $P(A_i = B_j)$ are the probability for Alice and Bob to have the same measure outcomes:

$$P(A_i = B_j) = \sum_{\ell=0}^{d-1} P(A_i = \ell, B_j = \ell), \quad (2)$$

and similarly

$$P(A_i = B_j + k) = \sum_{\ell=0}^{d-1} P(A_i = \ell, B_j = \ell + k \bmod d). \quad (3)$$

In general, one can show that for local theories only three of the four equalities in (2) can be satisfied at the same time, that is for local theories $I \leq 3$, while non-local theories could reach a value of $I = 4$.

Beside the generalization of the standard Bell inequality presented above, we also consider a second inequality, which at variance with the previous one is built with a number of terms that grows with the local dimension of the subsystems. Explicitly

$$I_d \equiv \sum_{k=0}^{[d/2]-1} \left(1 - \frac{2k}{d-1} \right) \times \left\{ \begin{aligned} &P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) \\ &+ P(A_2 = B_2 + k) + P(B_2 = A_1 + k) \\ &- [P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) \\ &+ P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1)] \end{aligned} \right\}. \quad (4)$$

As before, one can show that the maximal possible value achievable by this quantity is $I_d = 4$. However, LHV models allow only for $I_d \leq 2$.

Generalized maximally entangled (Bell) quantum states of two d -dimensional systems,

$$|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A \otimes |j\rangle_B, \quad (5)$$

violate these inequalities for any value of d [5]. It has been shown [5] that, for $|\Phi^+\rangle$, the observables that are suited to test the generalized Bell inequalities have associated eigenstates of the form

$$|k\rangle_{A,a} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp \left[i \frac{2\pi}{d} j(k + \alpha_a) \right] |j\rangle_A, \quad (6a)$$

$$|l\rangle_{B,b} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \exp \left[i \frac{2\pi}{d} j(-l + \beta_b) \right] |j\rangle_B. \quad (6b)$$

Here, $a, b = 1, 2$ with $\alpha_1 = 0, \alpha_2 = 1/2, \beta_1 = -\beta_2 = 1/4$ [5]. Projections of the state of the system onto such states always leads to a violation of I and I_d , independently of the particular value of d . This is shown in Fig. 1, where we report the exact numerical values for both I and I_d as a function of the spin quantum number l of the system. Note that, if the probabilities are computed according to the rules of quantum mechanics, the maximal value of I achievable for $d = 2$ is $I_{\max} = 2 + \sqrt{2} \approx 3.41$ [17] and that numerical data show that I monotonically decreases with d .

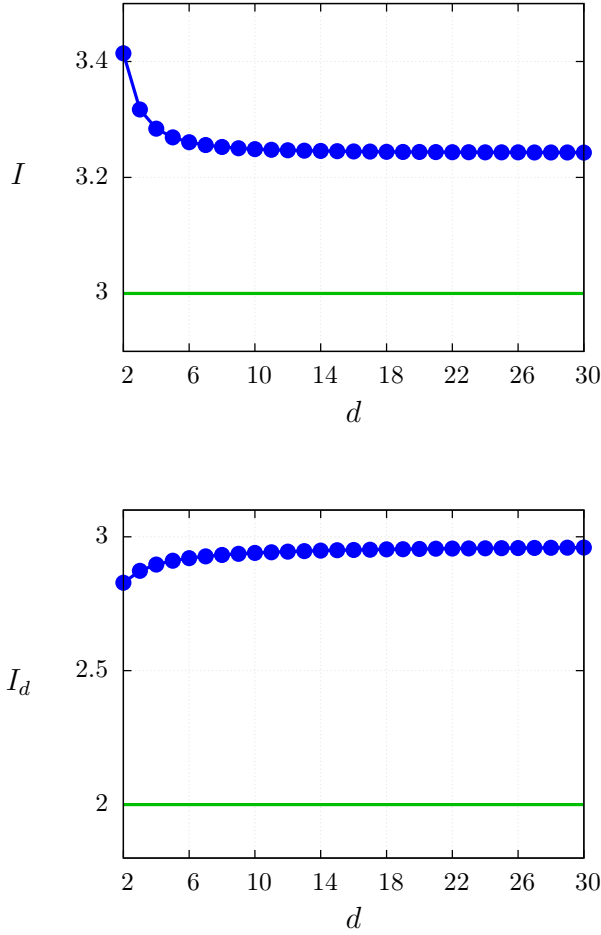


FIG. 1. (Color online) Values for the generalized Bell expressions I and I_d plotted against the spin quantum number $l = (d-1)/2$ of each subsystem. In order to evaluate I and I_d we have used the state $|\Phi^+\rangle$ and the measurement operators given in Eqs. (6). Bottom (green) lines: Maximum values achievable through local hidden variable models.

III. ROBUSTNESS OF GENERALIZED BELL INEQUALITIES

In this Section we characterize as a function of the system size the robustness of the above generalized Bell inequalities against perturbations of the projections identified in Eqs. (6). This is indeed a fundamental question to address when investigating the possibility to violate a Bell-like inequality using a macroscopic quantum system. In fact, any experimental test of this kind will have to cope with non-idealities in the identification of the most suitable projections to implement. Our study is thus aimed at determining the tolerance allowed by the Bell-like tests addressed above against imperfect measurement settings.

In order to attack this problem, we consider the equivalence between an improperly arranged measurement setting [i.e., projections of the ideal generalized Bell state performed on states that differ from those in Eqs. (6), but still being mutually orthogonal] and the performance of ideal measurements

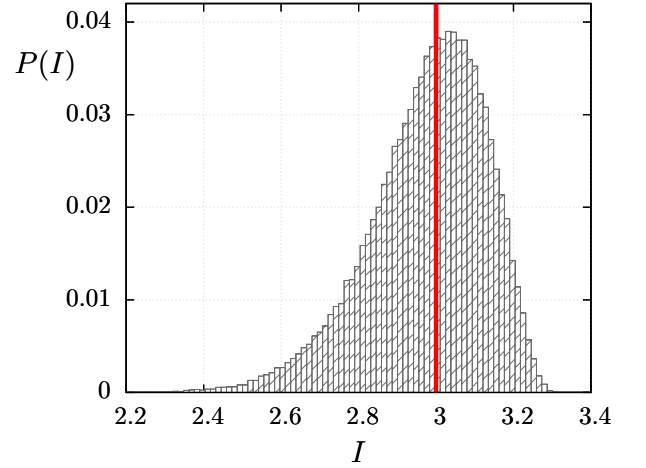


FIG. 2. (Color online) Histogram of the probabilities $P(I)$ to obtain the values I for the maximally entangled state given in Eq. (5) with $l = 1$ for a perturbation $\varepsilon \approx 0.233$. The red line $I = 3$ indicates the border between classical ($I \leq 3$) and non-classical correlations ($I > 3$).

on a state that is locally rotated by an unknown amount and around an unspecified direction. We thus perform a stability analysis in the regime of small perturbations by independently rotating each subsystem by an angle ε around a random axis. The rotations are performed by applying the bi-local unitary transformation

$$\hat{U} = \hat{U}_A \otimes \hat{U}_B = e^{i\varepsilon \hat{H}_A} \otimes e^{i\varepsilon \hat{H}_B}, \quad (7)$$

where \hat{H}_μ ($\mu = A, B$) is a d -dimensional Hermitian operator whose entries $(H_\mu)_{ij}$ are randomly chosen from a uniform distribution of values within the range $-1 \leq |(H_\mu)_{ij}| \leq 1$.

We have performed an extensive numerical analysis of the value of the Bell expressions I and I_d within the framework discussed above. In order to obtain the probability distributions $P(B)$ for $B = I, I_d$ we have accumulated statistics by means of $N = 10^5$ random rotations for each value of ε and d considered in our study. A typical example of the results of this analysis is reported, for $B = I$, in Fig. 2: the distribution $P(I)$ is a Gaussian-like distribution with a negative skew. In order to provide a quantitative characterization of the robustness of the Bell inequalities, we focus on the violation probability

$$\mathcal{P}_B = \int_{B_c}^{\infty} P(B') dB', \quad (8)$$

where $B_c = 2, 3$ for $B, B' = I_d, I$, respectively. Such quantity is the area of the distribution $P(B)$ at the rightmost side of the LHV value B_c .

We have studied the behavior of the the probability \mathcal{P}_B against the dimension of the problem at hand: in Fig. 3 (top panel) we plot \mathcal{P}_B as a function of the spin quantum number l ($d = 2l + 1$), for various choices of ε and for both I and I_d . We first note that for $l = 1/2$ the results for the two Bell expressions coincide. On the contrary, from $l = 1$ onwards, $\mathcal{P}_{I_d} > \mathcal{P}_I$, namely violation of the Bell inequality for I_d , which includes

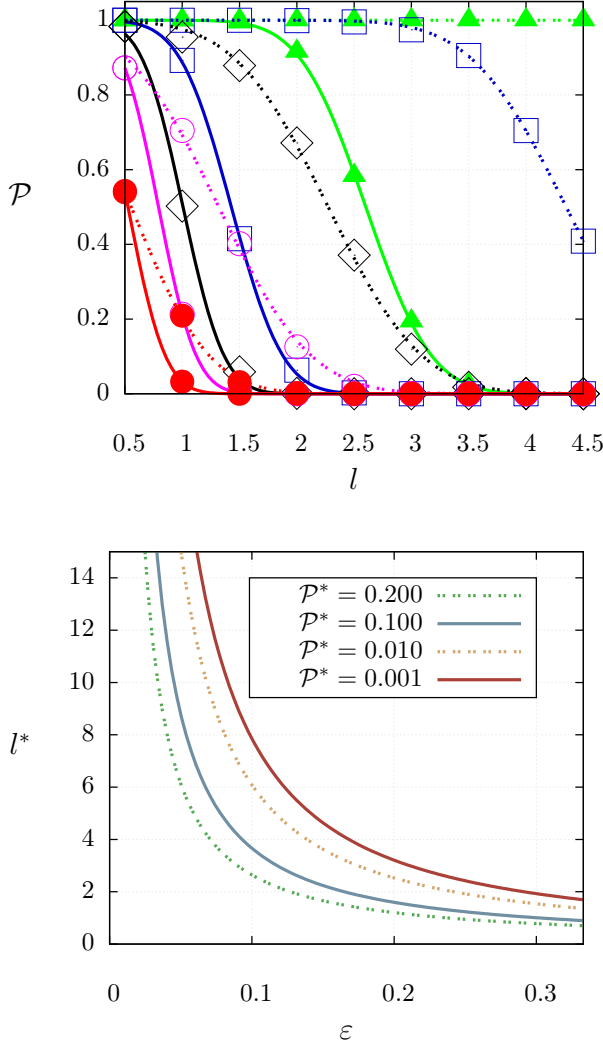


FIG. 3. (Color online) Upper panel: Violation probability \mathcal{P}_B as a function of the spin quantum number l for $\epsilon = 0.12, 0.18, 0.23, 0.29, 0.41$ (green triangles, blue squares, black diamonds, magenta circles, red filled circles). Solid and dashed curves show the fit (9) for I and I_d , respectively. Lower panel: estimate from Eq. (10) of the threshold l_B^* as a function of ϵ , for different threshold probabilities \mathcal{P}_B^* (see legend).

number of terms growing with the dimension of the system, is more resistant to local perturbations than violation for I . It is clear from the top panel of Fig. 3 that for all values of ϵ that we have considered and for a given minimum violation threshold \mathcal{P}_B^* , one can find a threshold value $l_B^*(\epsilon, \mathcal{P}_B^*)$ such that $\mathcal{P}_B < \mathcal{P}_B^*$ for all $l > l_B^*$. In order to quantify l_B^* , we fit the data of Fig. 3 (top panel) with the function

$$\frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{l - \bar{l}_B(\epsilon)}{\Delta_B(\epsilon)} \right) \right], \quad (9)$$

characterized by the mean value $\bar{l}_B(\epsilon)$ and the standard deviation $\Delta_B(\epsilon)$. This allows us to estimate l_B^* as

$$l_B^*(\epsilon, \mathcal{P}_B^*) \approx \bar{l}_B(\epsilon) + \Delta_B(\epsilon) \operatorname{erf}^{-1}(1 - 2\mathcal{P}_B^*), \quad (10)$$

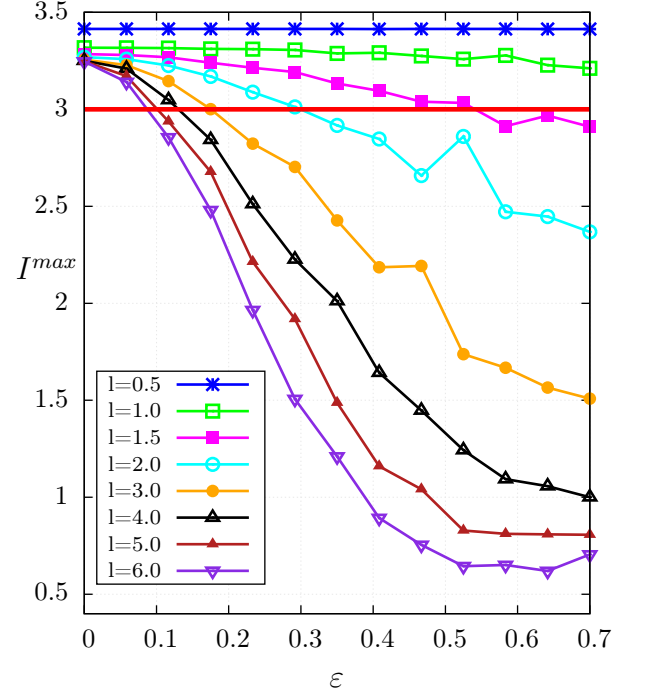


FIG. 4. (Color online) Maximal value $I_{\max}^{\max}(\epsilon)$ for different spin quantum number l . Red horizontal line indicates the violation bound $I = 3$.

with $\bar{l}_B(\epsilon) \approx 0.17\epsilon^{-0.61}$ and $\Delta_B(\epsilon) \approx 0.137\epsilon^{-1.38}$ for $B = I$; $\bar{l}_B(\epsilon) \approx 0.057\epsilon^{-2.52}$ and $\Delta_B(\epsilon) \approx 0.484\epsilon^{-0.50}$ for $B = I_d$. In Fig. 3 (bottom panel) we report l_B^* for several values of \mathcal{P}_B^* . These results clearly indicate that even for small error ϵ , the violation probability becomes rapidly negligible when increasing l .

Besides studying the probability to violate a given inequality, it is interesting to consider whether or not one is actually able to violate a Bell inequality in a d -dimensional system, for a given ϵ and a finite sampling. This is equivalent to estimate the quality of the statistical sampling of the tails of the Gaussian-like distribution $P(B)$. In order to do so, we adopt a pragmatic approach according to which we consider it possible to violate inequality for the Bell expression B if, with a sampling of $N = 10^5$ events, a single instance of violation is found. We then extract the maximum occurred value B^{\max} of the Bell expression and study it against ϵ for increasing system dimensions. Fig. 4 shows the results for the case of $B = I$, highlighting the existence of a value ϵ_I^* above which the corresponding inequality is no longer violated. Although the quantitative analysis is not reported here, we have found that an analogous critical value $\epsilon_{I_d}^*$ exists for the case of $B = I_d$, too.

The values of both ϵ_I^* and $\epsilon_{I_d}^*$ at given l are reported in Fig. 5, which also displays the comparison with the fitting function a/l^b , thus showing a power-law decay of the error threshold with the size of the system, with $b \approx -1.13$ for I and $b \approx -0.96$ for I_d . Consistently with our findings on the probability to violate a given inequality, we find that I_d is more robust than I as the corresponding critical value $\epsilon_{I_d}^*$ decays with l more slowly than ϵ_I^* .

We complete this Section by studying the stability of the generalized Bell inequalities when global perturbations are used instead of the bi-local ones considered so far. By repeating the same analysis illustrated above under the assumption that the random perturbation now acts on both the spins of our system, thus possibly deteriorating their shared correlations, we find the results reported by the green lines in Fig. 5. A global rotation of the state of the system results in a more pronounced decay of the threshold value of ϵ that, however, does not affect significantly the scaling with the size of the system. From the same power law fit a/l^b as above we obtain $b \approx -1.39$ for I and $b \approx -1.57$ for I_d . For random global rotations, it can be shown analytically (see appendix A) that the asymptotic (in d) decay of the violation probability \mathcal{P}_{I_d} is at least exponentially fast.

IV. RANDOM MEASUREMENT OPERATORS

In this section, we consider the violation of generalized Bell inequalities by a bipartite spin- d system prepared in an unknown state. In order to make our study more specific and concrete, we first consider a spin system prepared in a maximally entangled state but defined over random z -axes. In such a case, it is no longer possible to identify the form of the optimal measurement operators \hat{A} s and \hat{B} s, and the measurement settings given in Eqs. (6) would be unsuitable for the successful falsification of a given generalized Bell inequality. In order to address this point quantitatively, we resort to the observation that the picture given above is fully equivalent to that of a spin system prepared in the ideal generalized Bell state and subjected to unknown local measurement settings.

We thus perform measurements along random directions and check if this procedure allows to detect a violation with a statistically relevant probability. This procedure is not equivalent to the analysis we have done in the previous Section, as the *direction* of the projections entailed by the measurement steps needed for the construction of the Bell function are now fully random and not mutually orthogonal. Every measurement operator is rotated individually.

In Fig. 6 we report the resulting distributions of the values taken by I for various dimensions of the system. The distributions can be well fitted by Gaussian functions specified by mean value \bar{I} and standard deviation σ decreasing with l . Specifically, we find the best fit values $\bar{I} \propto l^{-0.79}$ and $\sigma \propto l^{-1.13}$. This implies that the probability of measuring a violation is as small as $\mathcal{P} \approx 4 \cdot 10^{-7}$ for $l = 1$, and is below machine precision for any $l > 1$. This is an interesting result, as it entails that, for systems with four or more local states, it is virtually impossible to violate a generalized Bell inequality, even when operating over a maximally entangled state but with random direction.

Given that random search fails to detect violation in general, we adopt a more sophisticated strategy and optimize the observables to be measured, employing the Nelder-Mead Simplex algorithm [18], which is a general-purpose gradient-free algorithm of vast use in complex multi-variable searches. We repeat the optimization from different randomly chosen initial

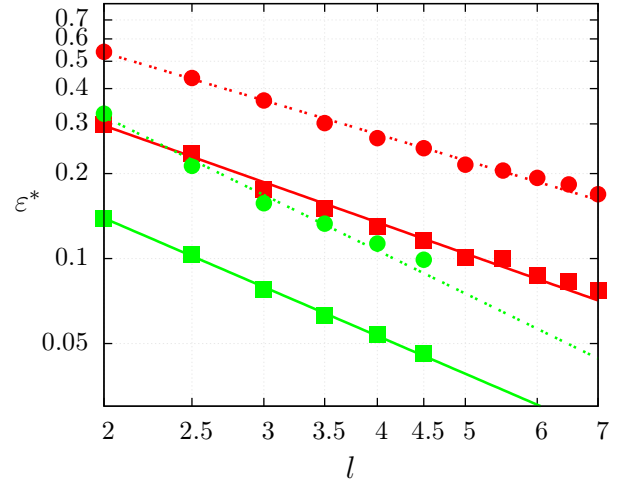


FIG. 5. Log-log plot of critical $\epsilon_B^*(l)$ with corresponding fits (form a/l^b) for $B = I$ (solid red with squares) and $B = I_d$ (dashed red with circles). The solid green line with squares denotes the global critical $\epsilon_{I,g}^*$ for global rotations for inequality I , the dashed green line with circles is $\epsilon_{I_d,g}^*$ for inequality I_d .

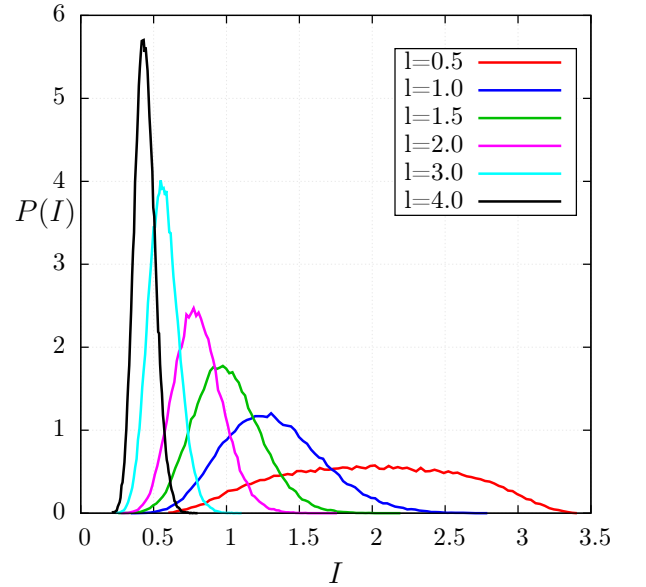


FIG. 6. (Color online) Histograms $P(I)$ of the Bell inequality I for Bell state $|\Phi^+\rangle$ and random measurement operators A_1, A_2, B_1 , and B_2 , for various spin numbers l .

sets of coefficients for the matrix representation of the observables A s and B s and report in Fig. 7 the maximum achieved value of I and I_d . As it can be seen, in both cases a violation of the Bell inequalities is found for systems of up to $l = 4.5$ and there is indication (as it should be) that for all system sizes it should be possible to find a set of observables capable to detect the violation. However, the computational time of one function evaluation scales as $O(l^6)$ for inequality I and as $O(l^7)$ for I_d [19]. In combination with an increasing number of function evaluations needed for increasing l , this once

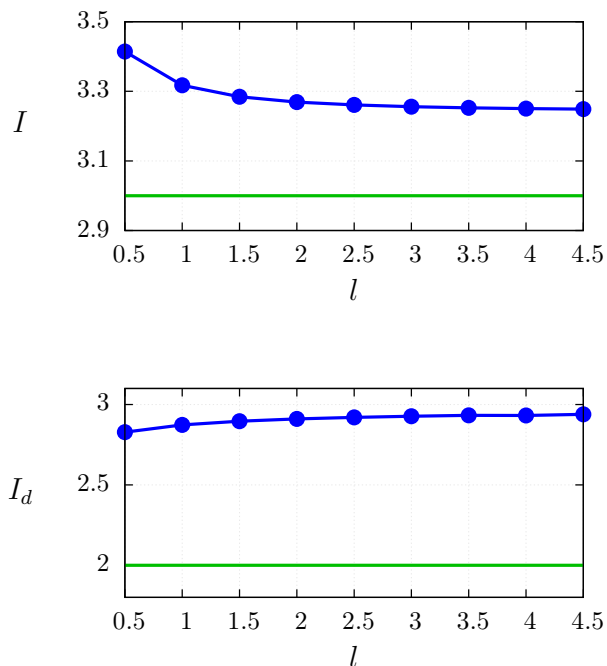


FIG. 7. (Color online) Optimized values I^{opt} (upper panel) and I_d^{opt} (lower panel) for different spin quantum numbers l (blue curve) and maximally entangled state $|\Phi^+\rangle$. In both panels the bottom (green) line corresponds to the border between classical and non-classical correlations.

more practically prevents the effectiveness of such protocol for increasing system sizes [20].

As a final step in our study, we now check how the violation of the generalized Bell inequalities behaves for random two-spin states. We considered two cases:

1. Random entangled states whose vectors have elements randomly chosen from a uniform distribution of values within the range $[-10, 10]$ and then properly normalized.
2. Product states of two random d -dimensional states (built up as described above).

For each value of l we consider five different random entangled states and two random product states and store the maximally violating instance for both cases. Typical results for maximally violating random entangled and a random product state are shown in Fig. 8, where we observe that violations are possible for both I and I_d and arbitrary values of l (blue circles). This of course is only true for random entangled states, as for random product states (red crosses) a violation is impossible. Nevertheless, the critical bounds can always be reached by classical random states. Note that once again, finding the proper setting to violate the Bell inequalities get harder for bigger system sizes: the cases for $l \geq 3.5$ in Fig. 8 display a lower violation with respect to the other ones. However, this is due only to the fact that we stop the optimization after one week of computational time, which in these cases is clearly not enough to achieve the maximal possible violation,

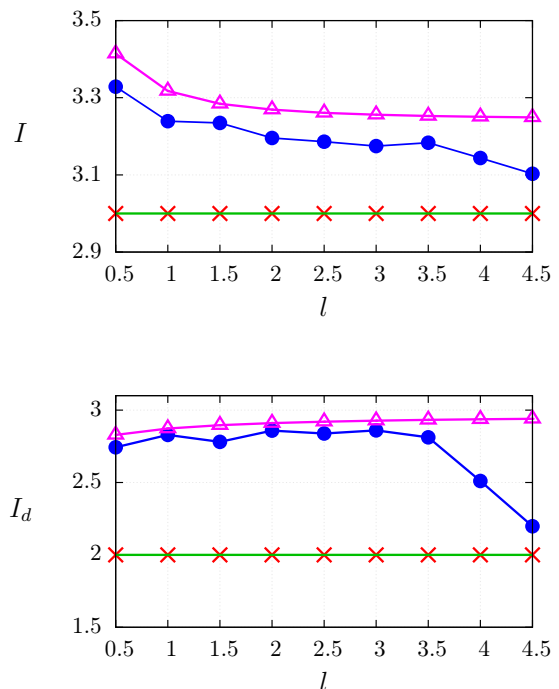


FIG. 8. (Color online) Optimal values for Bell expressions I (upper panel) and I_d (lower panel), for random entangled states (blue curve) and random separable states (crosses). Green line: bound for classical correlations, Triangles: values for I and I_d for the maximally entangled state and optimal measurement setting as reported in Fig. 1. Note: Blue points are always the maximal value of the five random states as described above.

demonstrating once more the increasing difficulties in finding a violation for increasing system size.

V. CONCLUSIONS

We have analyzed quantitatively the violation of Bell inequalities formulated for general d -dimensional systems. Our approach, which is oriented towards the provision of information relevant to experimentalists, was based on the careful evaluation of the impact that imperfections in both the state resource and the measurement settings could have on actually violating one of such generalized non-locality tests. We have found that both bi-local and global imperfections in the arrangements for the measurement settings needed to test a given inequality affect the quality of the corresponding Bell test in a system size-dependent fashion. Non-locality in bipartite systems spanning large Hilbert spaces is difficult to infer, even when managing maximally entangled states of two spin- l particles. Interestingly, we have identified size-dependent thresholds on the amount of noise that the system is able to tolerate before rendering the chance of violating a generalized Bell inequality negligibly small. We have also highlighted the influences due to the statistical sampling used to check the falsification of the Bell tests addressed in our work: while a uniform random sampling of the measurement sampling makes it

unlikely to single out an instance of violation even for qutrit systems, a more sophisticated statistical sampling reveals that it should be possible to find such instances regardless of the size of the local subsystems, although with increasing efforts being required.

We believe that this work helps addressing the inherent complexity of violating Bell inequalities above and beyond the usual studies used to assess their robustness, and provides useful information for any experimental endeavor aimed at testing non-local features in the state of systems spanning large Hilbert spaces, a situation that is quickly becoming more realistic and addressable. Furthermore, our work provides for a transparent interpretation of the emergence of classicality in terms of local realism at the macroscopic level: without resorting to decoherence mechanisms, the size-dependent noise thresholds we identified above describe the practical impossibility to detect non-classical behavior in large enough systems via Bell inequalities violations. In a word, in the unavoidable presence of noise, large systems will behave for all practical purposes classically under a Bell test.

Numerous avenues are left open by the questions addressed in this work, the most natural being whether the use of non-maximally entangled states leads to a less demanding inference of non-locality in large-sized systems [21]. In this context, it would be interesting to explore the robustness of the so-called *embezzling states* [22], i.e. of a family of partially entangled states which exhibit maximal violation of Bell inequalities, under the random errors addressed herein.

VI. ACKNOWLEDGEMENTS

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Appendix A: Asymptotic decay of \mathcal{P}_{I_d}

I_d , as defined in Eq. (4), is a function of the (pure) state ψ of the bipartite system, hence a function on the $2d^2 - 1$ dimensional real unit sphere S_{2d^2-1} . It will be shown that this function is subject to the “concentration of measure” phenomenon [23], meaning that, if ψ is chosen at random from the uniform distribution on S_{2d^2-1} , then the probability that $|I_d(\psi)|$ is larger than an arbitrarily small positive number is exponentially small with d ; notably,

$$\mathcal{P}\{|I_d(\psi)| \geq \varepsilon\} \leq 2 \exp\left(-\frac{d^2 \varepsilon^2}{192(6+d)\pi^3}\right). \quad (\text{A1})$$

To show this it is convenient to rewrite I_d in the form:

$$\begin{aligned} I_d(\psi) &= R_d(\psi) - S_d(\psi), \\ R_d(\psi) &= \sum_{1 \leq i, j \leq 2} \sum_{k=0}^{s-1} c(k) \|\hat{P}_k^{(i,j)} \psi\|^2, \\ S_d(\psi) &= \sum_{1 \leq i, j \leq 2} \sum_{k=0}^{s-1} c(k) \|\hat{P}_{-k-1}^{(i,j)} \psi\|^2, \end{aligned} \quad (\text{A2})$$

where $s = [d/2]$, $c(k) = 1 - 2k/(d-1)$. For $i, j = 1, 2$ and $-d+1 \leq k \leq d-1$, $\hat{P}_k^{(i,j)}$ is the projector onto the d -dimensional subspace wherein $A_i - B_j = n(i, j, k)$, where :

$$n(1, 1, k) = k, \quad n(1, 2, k) = -k,$$

$$n(2, 1, k) = -k - 1, \quad n(2, 2, k) = k.$$

The average of $I_d(\psi)$ with respect to the uniform, normalized measure on S_{2d^2-1} is 0, because projectors which share the same superscript i, j in R_d and in S_d are unitarily equivalent and the uniform measure is unitarily invariant. Then Levy’s lemma states that [23]

$$\mathcal{P}\{|I_d(\psi)| \geq \varepsilon\} \leq 2 \exp\left(-\frac{d^2 \varepsilon^2}{9\pi^3 \ell_d^2}\right), \quad (\text{A3})$$

where ℓ_d is the Lipschitz constant of I_d , to be presently estimated. First we write:

$$|I_d(\psi) - I_d(\phi)| \leq |R_d(\psi) - R_d(\phi)| + |S_d(\psi) - S_d(\phi)|, \quad (\text{A4})$$

and we note that, for an arbitrary projector \hat{P} and arbitrary normalized states ψ, ϕ

$$|\|\hat{P}\psi\|^2 - \|\hat{P}\phi\|^2| \leq 2\|\hat{P}(\psi - \phi)\|.$$

It follows that:

$$\begin{aligned} |I_d(\psi) - I_d(\phi)| &\leq 2 \sum_{1 \leq i, j \leq 2} \sum_{k=0}^{s-1} c(k) (\|\hat{P}_k^{(i,j)}(\psi - \phi)\| + \|\hat{P}_{-k-1}^{(i,j)}(\psi - \phi)\|). \end{aligned} \quad (\text{A5})$$

In the above sum, projectors $\hat{P}_k^{(i,j)}$ that share the superscript (i, j) with different k are mutually orthogonal, so $\hat{P}_k^{(i,j)} + \hat{P}_{-k-1}^{(i,j)}$ is again a projector $\hat{R}_k^{(i,j)}$, and

$$\|\hat{P}_k^{(i,j)}(\psi - \phi)\| + \|\hat{P}_{-k-1}^{(i,j)}(\psi - \phi)\| \leq \sqrt{2}\|\hat{R}_k^{(i,j)}(\psi - \phi)\|;$$

using this in (A5) we get

$$\begin{aligned} |I_d(\psi) - I_d(\phi)| &\leq 2\sqrt{2} \sum_{1 \leq i, j \leq 2} \sum_{k=0}^{s-1} c(k) \|\hat{R}_k^{(i,j)}(\psi - \phi)\| \\ &\leq 2\sqrt{2} \sum_{1 \leq i, j \leq 2} \left(\sum_{k=0}^{s-1} c(k)^2\right)^{1/2} \left(\sum_{k=0}^{s-1} \|\hat{R}_k^{(i,j)}(\psi - \phi)\|^2\right)^{1/2} \\ &\leq 8\sqrt{2}(1+s/3)^{1/2} \|\psi - \phi\|. \end{aligned} \quad (\text{A6})$$

In the last line, we used that $\hat{R}_k^{(i,j)}$ with the same i, j and different k are orthogonal. This inequality shows that $\ell_d \leq 8\sqrt{2}(1+s/3)^{1/2}$. Using this in Levy's lemma yields the announced estimate.

A sharper bound is obtained by using the following version of Levy's lemma:

$$\mathcal{P}\{|I_d - M_d| > \varepsilon\} \leq \exp(-d^2\varepsilon^2/(2\ell_d^2)), \quad (\text{A7})$$

where the median M_d is defined such that $\text{Prob}\{I_d \leq M_d\} =$

$1/2$. While we have no theoretical estimate for M_d , numerical results suggest that, at least with the observables we have chosen, the distribution of I_d is even. This would imply $M_d = 0$, and then:

$$\mathcal{P}\{|I_d| > \varepsilon\} \leq \exp(-3d^2\varepsilon^2/(128(6+d))).$$

While even this stricter bound becomes relevant only for dimensions $d \gtrsim 10$, bound (A1) proves rigorously that the violation probability \mathcal{P}_{I_d} , upper bounded by the right-hand side of (A1) for $\varepsilon = 2$, decays at least exponentially fast with d .

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